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Low Reynolds number flow in slowly varying axisymmetric tubes

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An asymptotic series solution is obtained for the low Reynolds number flow through an axisymmetric tube whose radius varies slowly in the axial direction. Expressions for the pressure drop along the tube and the shear stress at the wall are derived. The analysis is applicable to such problems as the flow through viscometric capillary tubes and the flow through blood vessels.

1. Introduction

To determine the shear stress distribution on the wall of an arteriosclerotic blood vessel, Lee & Fung (1970) obtained a numerical solution for the low Reynolds number flow through a locally constricted tube. Because of the inflexibility of numerical techniques, the shape of the tube was fixed during the study. It would seem to be of interest to find an analytic expression for the flow through an arbitrarily shaped tube. Towards this goal, the flow through a class of axisymmetric tubes of slowly varying radius is now considered.

The concept of a slowly varying flow forms the basis of a large class of problems in fluid mechanics for which viscous forces dominate the non-linear inertial forces (see, for example, Batchelor 1967, §4.8). Recently, this concept has been applied to the problem of peristaltic pumping by Burns & Parks (1967), Shapiro, Jaffrin & Weinberg (1969), and Lykoudis & Roos (1970). In these works, however, the inertial terms in the equations of motion are neglected and so the motion is approximated by a Stokes flow.

Tanner & Linnett (1965) extended the perturbation analysis of Blasius (1910) to predict the kinetic energy losses of viscometric capillary tubes. However, they neglected 'second-order' terms in the momentum equations so that the pressure was assumed to vary in the axial direction only. In the present work, a systematic analysis of the full equations of motion is undertaken and an asymptotic solution, valid for slowly varying tubes and low Reynolds number flows, is obtained.

2. Equations of motion

We consider the motion of a fluid in the domain D: $-\infty < x < \infty$, $0 < r < a(x)$ and $0 \leqslant \theta \leqslant 2\pi$, where (x, r, θ) are cylindrical polar co-ordinates such that $r = 0$ is the axis of symmetry for the duct and $r = a$ is the wall of the tube. By assuming that the motion is axisymmetric and steady, and that the fluid is incompressible, the equations for the conservation of momentum and volume are

$$\left.\begin{aligned} uu_x + vu_r + p_x/\rho &= \nu\{u_{xx} + (ru_r)_r/r\}, \\ uv_x + vv_r + p_r/\rho &= \nu\{v_{xx} + (rv_r)_r/r - v/r^2\}, \\ u_x + (rv)_r/r &= 0, \end{aligned}\right\} \qquad (2.1\,a, b, c)$$

where (u, v) are the fluid velocity components in the (x, r) directions, respectively; p is the pressure; ν is the constant fluid viscosity; ρ is the constant fluid density and the subscripts (x, r) denote partial differentiation with respect to (x, r), respectively.

We now introduce a stream function ψ such that

$$u = \psi_r/r \quad \text{and} \quad v = -\psi_x/r, \qquad (2.2)$$

and a vorticity component Ω such that

$$\Omega = u_r - v_x = \psi_{xx}/r + (\psi_r/r)_r. \qquad (2.3)$$

The pressure p may be eliminated from $(2.1\,a, b)$ to yield, from $(2.1\,b)$ to (2.3), the equation for the conservation of vorticity, viz.

$$\psi_r(\Omega/r)_x - \psi_x(\Omega/r)_r = \nu\{\Omega_{xx} + ((r\Omega)_r/r)_r\}. \qquad (2.4)$$

Thus, the motion within the domain D obeys equations (2.3) and (2.4).

On the boundaries of D, the following conditions must be satisfied:

$$\left.\begin{aligned} &u + (da/dx)\,v = 0 \quad \text{on} \quad r = a, \\ &\int_0^a dr \int_0^{2\pi} r\,d\phi\, u = 2\pi\psi_0, \quad \text{constant}, \\ \psi = 0 \quad \text{on} \quad r = 0, \quad &v = 0 \quad \text{on} \quad r = 0, \quad \partial u/\partial r = 0 \quad \text{on} \quad r = 0. \end{aligned}\right\} \qquad (2.5\,a\text{–}e)$$

Condition $(2.5\,a)$ states that there is no tangential fluid motion at the wall of the tube, whilst $(2.5\,b, c)$ specify the constant flow rate through the tube and $(2.5\,b)$ implies that there is no fluid motion normal to the tube wall. Conditions $(2.5\,d, e)$ ensure that the solution is regular at the axis of the tube. Putting (2.2) into (2.5), the boundary conditions become

$$\left.\begin{aligned} \psi_r = 0 \quad \text{on} \quad r = a(x), \quad \psi = \psi_0 \quad \text{on} \quad r = a(x), \quad \psi = 0 \quad \text{on} \quad r = 0, \\ \psi_x/r \to 0 \quad \text{as} \quad r \to 0, \quad (\psi_r/r)_r \to 0 \quad \text{as} \quad r \to 0. \end{aligned}\right\} \qquad (2.6\,a\text{–}e)$$

The required boundary-value problem is therefore completely specified by equations (2.3), (2.4) and (2.6).

We note that the solution domain D is infinite in the x direction and so an arbitrary 'initial' profile cannot be specified on some cross-section of the tube. Hence, the solution can be compared only with experimental results which are insensitive to the entry conditions of the tube.

3. Formulation for slowly varying tube

We now seek a solution of the system (2.3), (2.4) and (2.6) for the case when the radius of the tube a varies slowly in the axial direction. In particular, the function a is assumed to depend upon a small parameter ϵ such that

$$a(x;\epsilon) = a_0 S(\epsilon x/a_0), \quad (0 < \epsilon \ll 1), \tag{3.1}$$

where a_0 is a constant characteristic radius of the tube. In the limit $\epsilon \to 0$, the tube is of constant radius and the streamfunction is given by the familiar Poiseuille relation

$$(\psi)_{\epsilon=0} = \psi_0\{2(r/a_0)^2 - (r/a_0)^4\}; \tag{3.2}$$

i.e. the solution is independent of x. As ϵ increases from zero, we expect that the variation of ψ in the axial direction will depend upon ϵx rather than x alone. Thus, we look for a solution of the form

$$\psi = \psi_0 \phi(z, x^*;\ \epsilon), \quad \Omega = (\psi_0/a_0^3)\,\omega(z, x^*;\ \epsilon), \tag{3.3}$$

where $z = r/a_0$ is the normalized radial co-ordinate and $x^* = \epsilon x/a_0$ is a 'slowly varying' normalized axial co-ordinate. By putting (3.1) and (3.3) into (2.3)–(2.6), the normalized equations of motion become

$$\left.\begin{aligned} \omega &= (\phi_z/z)_z + \epsilon^2\phi_{x^*x^*}/z, \\ R\epsilon\{\phi_z(\omega/z)_{x^*} - \phi_{x^*}(\omega/z)_z\} &= ((z\omega)_z/z)_z + \epsilon^2\omega_{x^*x^*}, \end{aligned}\right\} \tag{3.4}$$

subject to the conditions

$$\left.\begin{aligned} \phi_z &= 0 \\ \phi &= 1 \end{aligned}\right\} \quad \text{on} \quad z = S(x^*), \tag{3.5}$$

$$\phi = 0 \quad \text{on} \quad z = 0, \qquad \phi_{x^*}/z \to 0 \quad \text{as} \quad z \to 0, \qquad (\phi_z/z)_z \to 0 \quad \text{as} \quad z \to 0, \tag{3.6}$$

where $R = \psi_0/a_0\nu$ is the characteristic Reynolds number of the flow.

The system (3.4)–(3.6) implies that, because $\epsilon = o(1)$, two distinct ranges exist for the order of magnitude of R; viz. (i) $\epsilon R = o(1)$ and (ii) $\epsilon R = O(1)$ or larger. Case (i) gives low Reynolds number ('viscous') solutions in which viscous effects dominate the non-linear inertial effects. In case (ii), the inertial terms are comparable with, or greater than, the viscous terms, and hence this gives high Reynolds number ('inertial') solutions. We shall consider only case (i); in particular, we take R equal to a constant which is formally of order unity. All the low Reynolds number solutions can readily be derived from this solution.

4. Low Reynolds number solution

Assuming that R is fixed and is of order unity, we expand ϕ and ω in asymptotic power series in ϵ, substitute into the system (3.4)–(3.6), and equate the coefficients of like powers of ϵ. That is,

$$\phi = \sum_{n=0}^{\infty} \epsilon^n \phi^{(n)}(z, x^*;\ R) \quad \text{and} \quad \omega = \sum_{n=0}^{\infty} \epsilon^n \omega^{(n)}(z, x^*;\ R), \tag{4.1}$$

where the coefficients $\phi^{(n)}$ and $\omega^{(n)}$ are determined successively from the following systems of equations:

$$\omega^{(0)} = (\phi_z^{(0)}/z)_z, \quad ((z\omega^{(0)})_z/z)_z = 0; \tag{4.2}$$

$$\omega^{(1)} = (\phi_z^{(1)}/z)_z, \ ((z\omega^{(1)})_z/z)_z = R\{\phi_z^{(0)}(\omega^{(0)}/z)_{x^*} - \phi_{x^*}^{(0)}(\omega^{(0)}/z)_z\}; \tag{4.3}$$

$$\omega^{(2)} = (\phi_z^{(2)}/z)_z + \phi_{x^*x^*}^{(0)}/z, \quad ((z\omega^{(2)})_z/z)_z = R\{\phi_z^{(0)}(\omega^{(1)}/z)_{x^*} + \phi_z^{(1)}(\omega^{(0)}/z)_{x^*} - \phi_{x^*}^{(0)}(\omega^{(1)}/z)_z - \phi_{x^*}^{(1)}(\omega^{(0)}/z)_z\} - \omega_{x^*x^*}^{(0)}; \tag{4.4}$$

etc. subject to the boundary conditions

$$\left.\begin{array}{l} \left.\begin{array}{l}\phi_z^{(n)} = 0 \\ \phi^{(0)} = 1, \quad \phi^{(n)} = 0 \quad \text{for} \quad n \neq 0\end{array}\right\} \quad \text{on} \quad z = S, \\ \phi^{(n)} = 0 \quad \text{on} \quad z = 0, \\ \left.\begin{array}{l}\phi_{x^*}^{(n)}/z \to 0 \\ (\phi^{(n)}/z)_z \to 0\end{array}\right\} \quad \text{as} \quad z \to 0. \end{array}\right\} \tag{4.5a–e}$$

The zeroth-order equations (4.2) may be readily solved to yield

$$\phi^{(0)} = A(z^*)z^4 + B(x^*)z^2 + C(x^*)z^2(\ln z - \tfrac{1}{2}) + D(x^*),$$

where A, B, C and D are functions of x^* which must be determined from the boundary conditions. Now conditions (4.5*c–e*) imply that C and D must be zero. The functions A and B are easily found from conditions (4.5*a, b*) so that finally

$$\phi^{(0)} = 2(z/S)^2 - (z/S)^4. \tag{4.6}$$

Comparing (4.6) with (3.2), we see that the zeroth-order solution is the Poiseuille flow appropriate to the local radius of the tube. Similarly, the first-order solution is found from (4.3), (4.5) and (4.6) to be

$$\phi^{(1)} = RS^4(1/S^4)_{x^*}\{\tfrac{1}{36}(z/S)^8 - \tfrac{1}{6}(z/S)^6 + \tfrac{1}{4}(z/S)^4 - \tfrac{1}{9}(z/S)^2\}. \tag{4.7}$$

This represents an inertial correction allowing for the advection of zeroth-order vortex lines by the zeroth-order axial velocity component. By putting (4.6) and (4.7) into (4.4) and (4.5), the second-order solution is found after much algebra to be given by

$$\phi^{(2)} = R^2 \sum_{n=1}^{6} (-)^{n+1} a_n (z/S)^{2n} + \tfrac{1}{6}S^6(1/S^4)_{x^*x^*}\{\tfrac{1}{2}(z/S)^6 - (z/S)^4 + \tfrac{1}{2}(z/S)^2\}, \tag{4.8}$$

where

$$a_1 = \tfrac{1}{900}\{\tfrac{497}{8}P + 13Q\}, \qquad a_2 = \tfrac{1}{720}\{\tfrac{11}{3}P + 29Q\},$$

$$a_3 = \tfrac{1}{24}\{\tfrac{2}{9}P + Q\}, \qquad a_4 = \tfrac{1}{144}\{P + 3Q\},$$

$$a_5 = \tfrac{1}{480}\{P + \tfrac{8}{3}Q\}, \qquad a_6 = \tfrac{1}{1800}\{\tfrac{1}{3}P + Q\},$$

and

$$P = S^8[(1/S^4)_{x^*}]^2, \quad Q = S^4(1/S^4)_{x^*x^*}.$$

The first terms in $\phi^{(2)}$ represent an inertial correction accounting for the advection and stretching of the zeroth- and first-order vortex lines by the first- and zeroth-order, respectively, velocity components. The second term in (4.8) is a viscous correction which accounts for the curvature in the axial direction of the zeroth-order vortex lines.

Thus, from (4.1) to (4.8), the normalized streamfunction ϕ and vorticity ω of the motion are given by

$$\phi = 2(z/S)^2-(z/S)^4-\frac{R\epsilon}{S}\frac{dS}{dx^*}\{\tfrac{1}{9}(z/S)^8-\tfrac{2}{3}(z/S)^6+(z/S)^4-\tfrac{4}{9}(z/S)^2\}$$
$$+R^2\epsilon^2\sum_{n=1}^{6}(-)^{n+1}a_n(z/S)^{2n}+\epsilon^2\left\{5\left(\frac{dS}{dx^*}\right)^2-S\frac{d^2S}{dx^{*2}}\right\}$$
$$\times\{\tfrac{1}{3}(z/S)^6-\tfrac{2}{3}(z/S)^4+\tfrac{1}{3}(z/S)^2\}+O(\epsilon^3), \tag{4.9}$$

and

$$\omega = -\frac{8}{S^3}(z/S)-\frac{8R\epsilon}{S^3}\frac{dS}{dx^*}\{\tfrac{2}{3}(z/S)^5-2(z/S)^3+(z/S)\}$$
$$+\frac{R^2\epsilon^2}{S^3}\sum_{n=2}^{6}(-)^{n+1}2n(2n-2)\,a_n(z/S)^{2n-3}+\epsilon^2S^3(1/S^4)_{x^*x^*}$$
$$\times\{(z/S)^3-\tfrac{4}{3}(z/S)\}+2\epsilon^2S(1/S^2)_{x^*x^*}(z/S)+O(\epsilon^3). \tag{4.10}$$

Equations (4.9)–(4.10) are valid for all Reynolds numbers such that $\epsilon R = o(1)$.

5. Pressure distributions

We have shown that the zeroth-order flow through an axisymmetric tube is a Poiseuille flow. Since, for a Poiseuille flow, the pressure is a linear function of the axial co-ordinate and it is inversely proportional to the Reynolds number, we seek a solution for the pressure of the form

$$p = (\rho\nu\psi_0/a_0^3)\,q, \tag{5.1}$$

where

$$q = xF(x^*)+P(z,x^*;\,\epsilon)$$

and

$$P = \sum_{n=0}^{\infty}\epsilon^n P^{(n)}(z,x;\,R).$$

By putting (2.2), (3.3), (4.1) and (5.1) into (2.1) and equating the coefficients of like powers of ϵ, a system of equations for the functions F and $P^{(n)}$ is found. The equations for F, $P^{(0)}$ and $P^{(1)}$ are

$$(x^*F)_{x^*} = (z(\phi_z^{(0)}/z)_z)_z/z; \tag{5.2}$$

$$\left.\begin{aligned}&P_z^{(0)} = 0,\\ &P_{x^*}^{(0)} = (z(\phi_z^{(1)}/z)_z)_z/z+R\{(\phi_{x^*}^{(0)}/z)(\phi_z^{(0)}/z)_z-(\phi_z^{(0)}/z)(\phi_z^{(0)}/z)_{x^*}\};\end{aligned}\right\} \tag{5.3a, b}$$

$$\left.\begin{aligned}&P_z^{(1)} = (\phi_{x^*}^{(0)}/z)/z^2-(z(\phi_{x^*}^{(0)}/z)_z)_z/z,\\ &P_{x^*}^{(1)} = (z(\phi_z^{(2)}/z)_z)_z/z+R\{(\phi_{x^*}^{(0)}/z)(\phi_z^{(1)}/z)_z+(\phi_{x^*}^{(1)}/z)(\phi_z^{(0)}/z)_z-(\phi_z^{(0)}/z)(\phi_z^{(1)}/z)_{x^*}\\ &\qquad -(\phi_z^{(1)}/z)(\phi_z^{(0)}/z)_{x^*}\}+(\phi_z^{(0)}/z)_{x^*x^*}.\end{aligned}\right\} \tag{5.4a, b}$$

Equations (5.2), (5.3*b*) and (5.4*b*) describe the forces against which the axial pressure gradient acts. To the zeroth order, there is only a viscous force proportional to the radial curvature of the axial velocity component. The first-order term includes an inertial force due to the rate of change of axial momentum and at the second order a viscous force proportional to the axial curvature of the axial velocity component is added. Equations (5.3a) and (5.4a) show that the radial pressure gradient is of order ϵ and that it balances a viscous force proportional to the radial curvature of the radial velocity component.

Equations (5.2)–(5.4) may be solved successively so that finally the normalized pressure distribution in the duct is found to be

$$q = -16\left\{\int^{x^*} \frac{dx^*}{\epsilon S^4(x_*)} + \frac{R}{4S^4} + \tfrac{11}{135}\epsilon R^2 \int^{x^*} \left(\frac{dS}{dx^*}\right)^2 \frac{dx^*}{S^6} + \tfrac{11}{180}\epsilon R^2 \frac{1}{S^5}\frac{dS}{dx^*} \right.$$
$$\left. + \tfrac{2}{3}\epsilon \int^{x^*} \left[5\left(\frac{dS}{dx^*}\right)^2 - S\frac{d^2S}{dx^{*2}}\right]\frac{dx^*}{S^4} + \epsilon\frac{1}{S^3}\frac{dS}{dx^*}[\tfrac{1}{2} + (z/S)^2] + O(\epsilon^2)\right\}. \quad (5.5)$$

The expression (5.5) is equal to the result obtained by Tanner & Linnett (1965) plus the last two terms which represent a second-order viscous correction.

6. Shear stress at wall of tube

The stress tensor for the motion is given by

$$\sigma_{ij} = -p\delta_{ij} + 2\rho\nu e_{ij}, \quad (6.1)$$

where the rate of strain components are

$$e_{xx} = \frac{\partial u}{\partial u}; \quad e_{rr} = \frac{\partial v}{\partial r}; \quad e_{xr} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial r}\right). \quad (6.2)$$

The shear stress at the wall $r = a(x)$ is

$$T = \left\{\sigma_{xr}\left[1 - \left(\frac{da}{dx}\right)^2\right] + (\sigma_{rr} - \sigma_{xx})\frac{da}{dx}\right\} \Big/ \left\{1 + \left(\frac{da}{dx}\right)^2\right\}. \quad (6.3)$$

Now from boundary conditions (2.6a)–(2.6b), it is seen that

$$\left.\begin{aligned} \psi_r &= 0 \\ \psi_{rr} &= -\psi_{xr}/(da/dx) \\ \psi_{xx} &= -\psi_{xr}/(da/dx) \end{aligned}\right\} \quad \text{on} \quad r = a(x). \quad (6.4a\text{–}c)$$

Putting (6.4) into (2.3), (6.1)–(6.3), we finally show that the shear stress at the wall is proportional to the vorticity; in particular,

$$T = \rho\nu\Omega \quad \text{on} \quad r = a(x). \quad (6.5)$$

We now introduce a normalized stress

$$\tau = (a_0^3 T)/(\rho\nu\psi_0), \quad (6.6)$$

and then, from (3.3), (4.10), (6.5) and (6.6), it is found that the normalized shear stress at the wall of the tube is given by

$$\tau = -\frac{8}{S^3}\left\{1 - \tfrac{1}{3}\epsilon R\frac{1}{S}\frac{dS}{dx^*} - \tfrac{67}{540}\epsilon^2 R^2 \frac{1}{S^2}\left(\frac{dS}{dx^*}\right)^2 \right.$$
$$\left. + \tfrac{1}{36}\epsilon^2 R^2 \frac{1}{S}\frac{d^2S}{dx^{*2}} - \tfrac{2}{3}\epsilon^2\left(\frac{dS}{dx^*}\right)^2 + \tfrac{1}{3}\epsilon^2 S\frac{d^2S}{dx^{*2}} + O(\epsilon^2)\right\}. \quad (6.7)$$

7. Discussion

Tanner & Linnett (1965) have investigated the flow through a capillary tube of exponentially increasing radius, i.e.

$$S = e^{x^*}, \tag{7.1}$$

in order to determine the Reynolds number at which separation occurs. Putting (7.1) into (6.7), we see that the shear stress at the wall of an exponential capillary tube is given by

$$\tau = -8e^{-3x^*}\{1 - \tfrac{1}{3}\epsilon R - \tfrac{13}{135}\epsilon^2 R^2 - \tfrac{1}{3}\epsilon^2 e^{2x^*} + O(\epsilon^3)\}. \tag{7.2}$$

Equation (7.2) suggests that separation, defined by $\tau = 0$, will occur at any Reynolds number provided that the tube is long enough; i.e. viscous effects will always cause separation eventually. However, for the tube used by Tanner & Linnett, the viscous term of order ϵ^2 was not large enough for this effect to be found.

Although (7.2) is valid only for $\epsilon R = o(1)$, an estimate of the Reynolds number at which separation occurs may be found by neglecting terms $O(\epsilon R)^n$, $n > 2$; viz. $\epsilon R \sim 2$. On the other hand, pressure loss measurements by Tanner & Linnett show that the low Reynolds number flow 'breaks down' for $\epsilon R \lesssim 1$. The reason for this is seen by calculating from (5.5) and (7.1) the pressure loss Δq between the sections 0 and x^* of the tube. Thus,

$$\Delta q = -(4/\epsilon)\{[1 - \epsilon R - \tfrac{22}{135}\epsilon^2 R^2 + O(\epsilon^3)][1 - e^{-4x^*}] - \epsilon^2[(z/S)^2 - \tfrac{5}{6}][1 - e^{-2x^*}] + O(\epsilon^3)\}. \tag{7.3}$$

Now (7.3) shows that the inertial forces cause the pressure loss to be less than the Poiseuille loss. In fact, the inertial pressure gain becomes comparable with the Poiseuille loss for $\epsilon R \sim 1$. Hence, the low Reynolds number flow régime breaks down at this stage, before separation has occurred.

We also note from (7.3) that the second-order viscous term acts to increase the pressure loss over most of the tube. However near the wall viscous forces tend to decrease the loss. This is consistent with the fact noted above that viscous effects lead to flow separation at the wall.

In a numerical study, Lee & Fung (1970) considered the flow through a locally constricted tube, defined by

$$S = 1 - \tfrac{1}{2}e^{-x^{*2}}, \tag{7.4}$$

where $\epsilon = 2$. Clearly, the asymptotic expansions derived above are not strictly valid for the large value of ϵ used in the Lee & Fung investigation. However, many features of the flow are adequately described by these expansions. Figure 1 compares the normalized shear stress at the wall for $R = 0$ as calculated from (6.7) neglecting terms $O(\epsilon^3)$ with that calculated exactly by Lee & Fung. It is seen that the maximum shear stress found from the approximate expression is only about 10 % greater than the exact result. Equation (6.7) shows that as the Reynolds number increases, the point of maximum shear stress moves upstream due to the slope of the tube wall; this effect was observed by Lee & Fung.

As noted above, a rough estimate of the conditions for flow separation may be found by considering the terms only $O(\epsilon)$ in the expression (6.7) for the shear stress. Then separation occurs when

$$\tfrac{1}{3}\epsilon R(1/S)\, dS/dx^* \sim 1.$$

Now for the locally constricted tube (7.4), the maximim value of $(1/3S)\, dS/dx^*$ is about 0·2, and hence separation should occur when $R \sim 2{\cdot}5$. This may be compared with the value of $R = 2{\cdot}475$ found by Lee & Fung.

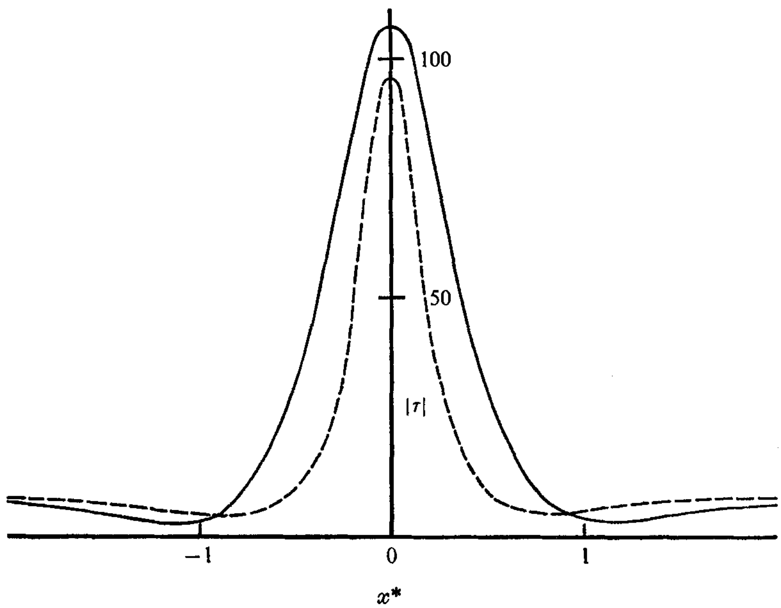

FIGURE 1. Normalized shear stress at the wall of a locally constricted tube (7·4) for $R = 0$; ———, from (6.7) to $O(\epsilon^2)$; ----, from Lee & Fung (1970).

Fry (1968) used the expression

$$\tau = \epsilon(S/2)\, \partial q/\partial x^* \quad \text{at} \quad z = S \tag{7.5}$$

to calculate the shear stress from the measured pressure gradient. Differentiating (5.5), we find that

$$\epsilon\frac{S}{2}\frac{\partial q}{\partial x^*} = -\frac{8}{S^3}\left\{1 - \epsilon R\frac{1}{S}\frac{dS}{dx^*} + O(\epsilon^2)\right\};$$

i.e. the term of order ϵ on the right-hand side of (7.5) is three times greater than that on the left-hand side (see (6.7)). Thus, the expression (7.5) is invalid.

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